

Finite blocklength converse bounds for quantum channels

William Matthews

Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB England

Stephanie Wehner

Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore, 117543

We derive upper bounds on the rate of transmission of classical information over quantum channels by block codes with a given blocklength and error probability, for both entanglement-assisted and unassisted codes, in terms of a unifying framework of quantum hypothesis testing with restricted measurements. Our bounds do not depend on any special property of the channel (such as memorylessness) and generalise both a classical converse of Polyanskiy, Poor, and Verdú as well as a quantum converse of Renner and Wang, and have a number of desirable properties. In particular our bound on entanglement-assisted codes is a semidefinite program and for memoryless channels its large blocklength limit is the well known formula for entanglement-assisted capacity due to Bennett, Shor, Smolin and Thapliyal.

I. INTRODUCTION

This work is concerned with the transmission of classical information over quantum channels by means of block codes. This is a central subject of study in quantum information theory, and the *asymptotic* rates of transmission for various types of code and channel in the large blocklength limit are the subject of celebrated theorems and intriguing open problems. A more fundamental problem, of both theoretical and practical interest, is to obtain upper (or *converse*) and lower (or *achievability*) bounds on the optimal transmission rate for a given error probability ϵ and *finite* blocklength n . As Figure 1 illustrates, finite blocklength effects can be substantial.

Without assumptions on the structure of the operation implemented by n channel uses (e.g. independence), there is only a notational difference between coding for n uses of a channel and coding for one use of a larger, composite channel, and so bounds which apply in this scenario are also known as ‘one-shot’ bounds. These are the subject of a number of recent results in quantum information [1–5] and remain an active topic of research in classical information [6, 7]. All bounds referred to in the remainder of this introduction are of this type.

Mosonyi and Datta [1], Wang and Renner [2] and Renes and Renner [3] have given converse and achievability bounds for classical-quantum channels. These can be applied to unassisted coding over general quantum channels by maximising the bound for the classical-quantum channel induced by a particular choice of encoding over all such choices. In [4] Datta and Hsieh derive converse and achievability results for *entanglement-assisted coding* over quantum channels in terms of smoothed min- and max-entropies.

The present work was inspired by a converse for classical channels (hereafter the ‘PPV converse’) given in a paper by Polyanskiy, Poor and Verdú [6] and further investigated by Polyanskiy [7] and Matthews [8]. We provide converses for both entanglement-assisted and unassisted coding over quantum channels, in terms of a quantum

hypothesis testing problem (illustrated in Fig. 3) with restrictions on the measurements for the unassisted case.

This provides a unifying framework which not only includes the PPV converse (for channels with finite input/output alphabets) but also the existing converse of Wang and Renner [2] (as applied to general quantum channels) as a special case.

Our converse for entanglement-assisted codes has some good properties. For a fixed blocklength, the converse of Datta and Hsieh [4] is unbounded as $\epsilon \rightarrow 0$ whereas our bound is a decreasing function of ϵ . It has a formulation as a semidefinite program, and its computation can be simplified (sometimes greatly) by using any symmetries the channel may possess. As an example, we show how to compute our bound exactly for n uses of a depolarising channel (see Section III and Fig. 1).

We give more than one converse for unassisted codes, which have various advantages and disadvantages. The next subsection describes the channel coding problem and gives a summary of our results.

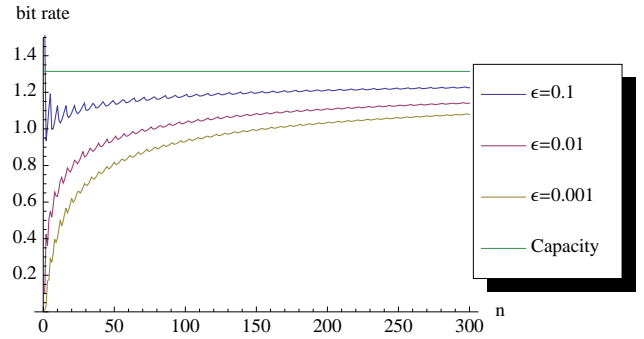


FIG. 1: Our upper bound (4) on the rate of entanglement-assisted codes evaluated for three different error probabilities ϵ , for the qubit depolarising channel with failure probability 0.15. The red line marks the capacity of the channel (roughly 1.31 bits/channel use) as given by the formula of Bennett, Shor, Smolin and Thapliyal [9].

A. Problem formulation and summary of results

As usual, a quantum system Q is associated with a Hilbert space \mathcal{H}_Q . By the dimension of the system $\dim(Q)$ we mean the dimension of the associated space, and this work deals only with finite-dimensional systems. By a *state* of Q we mean a density (i.e. positive, trace one) operator on \mathcal{H}_Q . We denote the set of all such density operators by $\mathcal{D}(\mathcal{H}_Q)$.

By an *operation* with input system A and output system B we mean a linear map from $\mathcal{D}(\mathcal{H}_A)$ to $\mathcal{D}(\mathcal{H}_B)$ which is completely positive and trace preserving. We denote the set of all such operations by $\mathbf{ops}(A \rightarrow B)$.

It is convenient to assume that every quantum system Q comes equipped with a canonical orthonormal basis which we call the *classical* basis, and whose members we denote by $|i\rangle_Q$ for $i = 1, \dots, \dim(Q)$.

As usual, a use (or uses) of a quantum channel with input system A and output system B is represented by an operation $\mathcal{E}_{B|A} \in \mathbf{ops}(A \rightarrow B)$.

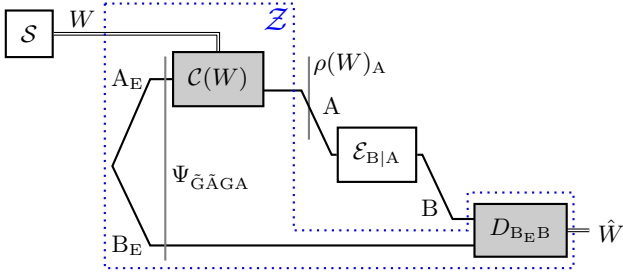


FIG. 2: An entanglement-assisted code \mathcal{Z} transmitting a message W chosen by a source \mathcal{S} , via a channel use \mathcal{E} . The average channel input induced by the source and encoding is $\rho_A = \sum_{w=1}^M \mathcal{S}(w) \rho(w)_A$.

Definition 1. In an **entanglement-assisted code** of size M , the sender and receiver have systems A_E and B_E in an entangled state $\Psi_{A_E B_E}$, and for each message $w \in \{1, 2, \dots, M\}$ there is an encoding operation $\mathcal{C}(w)_{A|A_E} \in \mathbf{ops}(A_E \rightarrow A)$. Following the use(s) of the channel, the decoder performs a POVM $D_{B|B}$ on $B_E B$ to obtain the decoded message.

Definition 2. An **unassisted code** can be viewed as a degenerate case where the decoding measurement operates only on the channel output B . Since B_E is completely ignored, there is no loss of generality if we take B_E and A_E to be trivial, one-dimensional systems. Then $\mathcal{C}(w)$ is completely specified by its constant output $\rho(w)_A$ on A .

Figure 2 illustrates an entanglement-assisted code \mathcal{Z} transmitting a message W chosen by a source \mathcal{S} via a channel use $\mathcal{E}_{B|A}$. The message W and the outcome \hat{W} of the decoding POVM are classical random variables. The source is specified by the probabilities

$$\mathcal{S}(w) := \Pr(W = w | \mathcal{S}).$$

The probability of error (which depends on the source, code and channel) is

$$\Pr(\hat{W} \neq W | \mathcal{E}, \mathcal{Z}, \mathcal{S}).$$

For an integer M let \mathcal{S}_M denote a source with M equiprobable messages i.e. $\mathcal{S}_M(w) = 1/M$.

Definition 3. We call a size M code \mathcal{Z} an (M, ϵ, ρ_A) code for \mathcal{E} , if $\Pr(\hat{W} \neq W | \mathcal{E}, \mathcal{Z}, \mathcal{S}_M) \leq \epsilon$, and its average input state is ρ_A .

We denote by $M_\epsilon^{\mathbf{C}}(\mathcal{E}_{B|A}, \rho_A)$ the largest M such that there is an (M, ϵ, ρ_A) code in class \mathbf{C} for $\mathcal{E}_{B|A}$. If $\mathbf{C} = \mathbf{E}$ then the codes can be entanglement-assisted; if it is omitted, we only allow unassisted codes.

Remark 4. Clearly, the largest size of any code in \mathbf{C} with error probability ϵ (which we denote by $M_\epsilon^{\mathbf{C}}(\mathcal{E}_{B|A})$) is just $\max_{\rho_A \in \mathcal{D}(\mathcal{H}_A)} M_\epsilon^{\mathbf{C}}(\mathcal{E}_{B|A}, \rho_A)$.

In [6] Polyanskiy, Poor and Verdú showed that many existing *classical* converse results can be easily derived from a finite blocklength converse which is obtained by a simple and conceptually appealing argument relating coding to hypothesis testing on the joint distribution of the channel input and channel output.

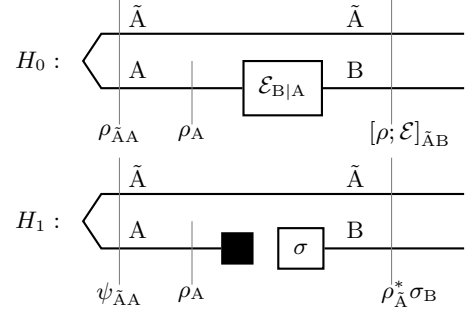


FIG. 3: The quantum hypothesis testing problem which appears in our bounds.

Our key results (Theorem 14 and Theorem 15) are generalisations of the PPV bound to quantum channels.

As shown in Fig. 3 the hypotheses specify quantum states of a bipartite system $\tilde{A}B$, where B is the output system of $\mathcal{E}_{B|A}$ and \tilde{A} is isomorphic to its input system. Define a canonical purification of the average input state ρ_A by $\psi_{\tilde{A}A} := \rho_A^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_A^{\frac{1}{2}}$, where $\Phi_{\tilde{A}A} := \sum_{i,j=1}^{\dim(A)} |i\rangle_{\tilde{A}} \langle i|_A \langle j|_{\tilde{A}} \langle j|_A$. Hypothesis H_0 is that $\tilde{A}B$ is in the state $[\rho; \mathcal{E}]_{\tilde{A}B} := \mathcal{E}_{B|A}[\psi_{\tilde{A}A}]$ obtained by acting on this purification with the channel $\mathcal{E}_{B|A}$, whereas hypothesis H_1 is that the state of system B has been replaced by σ_B , resulting in the product state $\rho_{\tilde{A}}^* \sigma_B$.

In section II we obtain converses for both entanglement-assisted and unassisted codes as a function of the minimum type II error $\beta := \Pr(\text{accept } H_0 | H_1, T)$ (i.e. the probability that the test T incorrectly accepts H_0 when actually H_1 was true), for tests T which have

type I error $\alpha := \Pr(\text{accept } H_1 | H_0, T)$ no greater than ϵ , and which can be implemented by operations in a class Ω which depends on the class of codes:

$$D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B) \quad (1)$$

$$= -\log \beta_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B}, \rho_A^* \sigma_B) \quad (2)$$

$$= \min_{T_{\tilde{A}B} \in \mathbf{T}^\Omega(\tilde{A}, B)} \{ \text{Tr} T_{\tilde{A}B} \rho_A^* \sigma_B : \text{Tr} T_{\tilde{A}B} [\rho; \mathcal{E}]_{\tilde{A}B} \geq 1 - \epsilon \} \quad (3)$$

where $\mathbf{T}^\Omega(\tilde{A}, B)$ is the set of all tests (POVM elements) on the system $\tilde{A}B$ which can be implemented by operations in the class Ω , (what this means is defined in Section IC). The class **ALL** only demands that $T_{\tilde{A}B}$ be a valid POVM element ($0 \leq T_{\tilde{A}B} \leq \mathbb{1}$). By generalising the construction in [6] which takes codes for classical channels to classical hypothesis tests to a construction which takes entanglement-assisted codes to quantum tests (i.e. POVM elements), we show that $\log M_\epsilon^\mathbf{E}(\mathcal{E}_{B|A}, \rho_A) \leq \min_{\sigma_B} D_\epsilon([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B)$, and hence that

$$\log M_\epsilon^\mathbf{E}(\mathcal{E}_{B|A}) \leq \max_{\rho_A} \min_{\sigma_B} D_\epsilon([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B) \quad (4)$$

(where the omission of the test class superscript means that it is **ALL**). When the channel is classical this bound reduces to the PPV converse.

The class **L** of *local* tests corresponds operationally to those which can be implemented by classical hypothesis testing on the joint outcome of local measurements (possibly correlated by shared randomness) on the two subsystems. Since our construction is shown to take unassisted codes to local tests, we obtain the bounds $\log M_\epsilon(\mathcal{E}_{B|A}, \rho_A) \leq \min_{\sigma_B} D_\epsilon^\mathbf{L}([\rho; \mathcal{E}]_{\tilde{A}B}, \rho_A^* \sigma_B)$ and

$$\log M_\epsilon(\mathcal{E}_{B|A}) \leq \max_{\rho_A} \min_{\sigma_B} D_\epsilon^\mathbf{L}([\rho; \mathcal{E}]_{\tilde{A}B}, \rho_A^* \sigma_B). \quad (5)$$

The bound (4) for entanglement-assisted codes has a number of desirable properties:

1. It is asymptotically tight for memoryless channels. In common with the bound of Datta and Hsieh [4], analysing the large block length behaviour of the bound for memoryless channels recovers the converse part of the single-letter formula for entanglement-assisted capacity proven by Bennett, Shor, Smolin and Thapliyal [9], as we show in subsection III D.
2. Generalising results of Polyanskiy [7] we show that $D_\epsilon([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B)$ is concave in ρ_A and convex in σ_B . This enables one to use symmetries of the channel to restrict the optimisation over ρ_A and σ_B to states with corresponding symmetries, as we show in subsection III E.
3. In subsection III B, we give an explicit formulation of the bound as *semidefinite program* (SDP) which is a natural generalisation of the linear program (LP) given in [8] for the PPV converse.

In subsection III C, the Wang-Renner bound is shown to be equivalent to making the (sometimes suboptimal [7]) choice $\sigma_B = \mathcal{E}_{B|A}[\rho_A]$ and taking Ω to be the class of operations **LC1** which can be implemented by local operations and one-way classical communication from Alice and Bob.

Since the Wang-Renner bound is asymptotically tight for the unassisted capacity (and even for the product state capacity, thus recovering the HSW theorem), the stronger bound (5) also has these properties, but is otherwise less attractive, as it lacks an SDP formulation and does not possess the convexity property mentioned above [10]. However, the formulation in terms of restricted hypothesis testing makes it clear that by moving to less restrictive conditions on the test, we might obtain weaker, but more tractable bounds.

When Ω is **LC1**, or the larger class **PPT** (of operations which remain completely positive when preceded and followed by partial transposition [11]), the convexity property does hold (see Theorem 19), as does the symmetrisation argument.

For **PPT** the bound is given by an SDP (see subsection III B). It seems unlikely that the **PPT** bound is in general, asymptotically tight, but it might prove useful for certain channels.

To demonstrate the use of our bound on entanglement-assisted codes, we show in section IV how to evaluate it exactly for depolarising channels. We also discuss the relationship of the work to existing results on strong converse bounds for quantum channels, and to security proofs in the noisy-storage model (section V).

B. Classes of operation on bipartite systems

By a *sub-operation* we mean a trace non-increasing completely positive map. Let $\mathbf{subops}(A \rightarrow A')$ denote the set of all sub-operations taking states of the system A to states of the system A' .

Similarly, let $\mathbf{subops}^\Omega(A \rightarrow A', B \rightarrow B')$ ($\mathbf{ops}^\Omega(A \rightarrow A', B \rightarrow B')$) denote the set of all sub-operations (operations) taking states of the bipartite system AB to states of $A'B'$, which belong to class Ω .

A sub-operation $\mathcal{L}_{A'B'|AB} \in \mathbf{subops}(A \rightarrow A', B \rightarrow B')$ belongs to the class **PPT** if it is positive-partial-transpose preserving, i.e. if $t_{B'} \circ \mathcal{L} \circ t_B$ is completely positive, where t denotes the transpose map. $\mathcal{L}_{A'B'|AB}$ belongs to **LC1** if it can be implemented by local operations and one-way classical communication from Alice to Bob. This means it can be written in the form

$$\mathcal{L}_{A'B'|AB} = \sum_a \mathcal{F}_{A'|A}^{(a)} \mathcal{D}_{B'|B}^{(a)}$$

where $\sum_a \mathcal{F}_{A'|A}^{(a)} \in \mathbf{subops}(A \rightarrow A')$ and $\mathcal{D}_{B'|B}^{(a)} \in \mathbf{subops}(B \rightarrow B')$ for each a . Throughout, we will omit tensor products if it is clear which system the operations act on. $\mathcal{L}_{A'B'|AB}$ belongs to **L** if it can be implemented

by local operations and shared randomness, which means it can be written

$$\mathcal{L}_{A'B'|AB} = \sum_r p_r \mathcal{F}_{A'|A}^{(r)} \mathcal{D}_{B'|B}^{(r)}$$

where $\mathcal{F}_{A'|A}^{(r)} \in \mathbf{subops}(A \rightarrow A')$ and $\mathcal{D}_{B'|B}^{(r)} \in \mathbf{subops}(B \rightarrow B')$. These classes are all closed under composition of compatible sub-operations. They are also closed under convex combination. Furthermore, they form a hierarchy $\mathbf{PPT} \supset \mathbf{LC1} \supset \mathbf{L}$ [11, 12].

Let $\mathbf{M}^\Omega(A, B)$ denote the set of all operations $\mathcal{M}_{C|AB} \in \mathbf{ops}(AB \rightarrow C)$ of the form

$$\mathcal{M}_{C|AB} = \sum_{k=1}^{\dim(C)} |k\rangle\langle k|_C \text{Tr}_{A'B'} \mathcal{L}_{A'B'|AB}^{(k)}$$

where C is a finite-dimensional system acting as a classical register, and where each $\mathcal{L}_{A'B'|AB}^{(k)} \in \mathbf{subops}^\Omega(A \rightarrow A', B \rightarrow B')$.

C. Quantum Hypothesis Testing with Restricted Measurements.

In a classical hypothesis testing problem (with simple hypothesis) there are two hypotheses H_0 and H_1 of the form is ‘The random variable R has distribution $P^{(i)}$ ’. A statistical test T can be specified by the giving the probabilities $T(r) = \Pr(\text{accept } H_0 | T, R = r)$.

The ‘type-I error’ of T is

$$\alpha(P^{(0)}, T) = \Pr(\text{accept } H_1 | H_0, T) \quad (6)$$

$$= 1 - \sum_r P^{(0)}(r) T(r) \quad (7)$$

while the ‘type-II error’ of T is

$$\beta(P^{(1)}, T) = \Pr(\text{accept } H_0 | H_1, T) \quad (8)$$

$$= \sum_r P^{(1)}(r) T(r). \quad (9)$$

Definition 5.

$$\beta_\epsilon(P^{(0)} \| P^{(1)}) := \min \beta(P^{(1)}, T) \quad (10)$$

subject to

$$\alpha(P^{(0)}, T) \leq \epsilon, \quad (11)$$

$$\forall r : 0 \leq T(r) \leq 1. \quad (12)$$

In a quantum hypothesis testing problem (with simple hypothesis) there are two hypotheses H_0 and H_1 of the form is ‘The state of system Q is $\tau_Q^{(i)}$ ’. In order to distinguish between these situations it is necessary to perform a measurement Q , the outcome of which is then subjected to a classical hypothesis test.

We extend Definition 5 to the quantum case, first to $\tau^{(i)}$ both diagonal in the classical basis by the trivial identification of such states with distributions, and then to the general case by

Definition 6.

$$\beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) := \inf \beta_\epsilon(\mathcal{M}_{C|AB}[\tau_{AB}^{(0)}] \| \mathcal{M}_{C|AB}[\tau_{AB}^{(1)}]) \quad (13)$$

subject to

$$\mathcal{M}_{C|AB} \in \mathbf{M}^\Omega(A, B). \quad (14)$$

$$\text{and } D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) := -\log \beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}).$$

A trivial but very useful result is the following generalised data processing inequality.

Proposition 7 (Data processing inequality). *If Ω is closed under composition, then for any operation $\mathcal{N}_{A'B'|AB} \in \mathbf{ops}^\Omega(A \rightarrow A', B \rightarrow B')$*

$$D_\epsilon^\Omega(\mathcal{N}_{A'B'|AB}[\tau_{AB}^{(0)}] \| \mathcal{N}_{A'B'|AB}[\tau_{AB}^{(1)}]) \leq D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}).$$

Corollary 8. If there is also an operation $\mathcal{N}'_{AB|A'B'} \in \mathbf{ops}^\Omega(A' \rightarrow A, B' \rightarrow B)$ such that $\mathcal{N}'_{AB|A'B'} \circ \mathcal{N}_{A'B'|AB}[\tau_{AB}^{(0)}] = \tau_{AB}^{(0)}$ and $\mathcal{N}'_{AB|A'B'} \circ \mathcal{N}_{A'B'|AB}[\tau_{AB}^{(1)}] = \tau_{AB}^{(1)}$ then

$$D_\epsilon^\Omega(\mathcal{N}_{A'B'|AB}[\tau_{AB}^{(0)}] \| \mathcal{N}_{A'B'|AB}[\tau_{AB}^{(1)}]) = D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}).$$

Since the states of C in Definition 6 are classical, there is an optimal test which measures in the classical basis. Supposing that it has probability p_k of accepting H_0 when it measures k , then the probability of the test accepting H_0 when the state of AB is τ_{AB} is

$$\sum_k p_k \text{Tr}_{A'B'} \mathcal{L}_{A'B'|AB}^{(k)}[\tau_{AB}] = \text{Tr}_{A'B'} \mathcal{L}_{A'B'|AB}[\tau_{AB}] \quad (15)$$

$$= \text{Tr}_{AB} T_{AB}[\tau_{AB}] \quad (16)$$

where $\mathcal{L} = \sum_k p_k \mathcal{L}^{(k)} \in \mathbf{subops}^\Omega(A \rightarrow A', B \rightarrow B')$ by convexity, and $T_{AB} = \sum_i M(i)^\dagger M(i)$ where $M(i)$ are Kraus elements for \mathcal{L} . T_{AB} is the POVM element (which we call simply a ‘test’), of the outcome of a measurement on AB which can be implemented by operations in Ω which corresponds to the acceptance of hypothesis H_0 . We denote the set of such tests on AB by $\mathbf{T}^\Omega(A, B)$.

Proposition 9.

$$\beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) := \inf \text{Tr}_{\tau_{AB}^{(1)}} T_{AB} \quad (17)$$

subject to

$$\text{Tr} \tau^{(0)} T_{AB} \geq 1 - \epsilon, \quad (18)$$

$$T_{AB} \in \mathbf{T}^\Omega(A, B). \quad (19)$$

Proposition 10. $T_{AB} \in \mathbf{T}^{\mathbf{PPT}}(A, B)$ if and only if $0 \leq T_{AB} \leq \mathbb{1}_{AB}$ and $0 \leq \mathfrak{t}_B[T_{AB}] \leq \mathbb{1}_{AB}$ where \mathfrak{t}_B is the transposition map on system B . (See [13].)

Remark 11. For the classes of operations considered in this paper (**ALL**, **PPT**, **LC1** and **L**), $\mathbf{T}^\Omega(A, B)$ is a closed set, and so for these classes the infimum in Proposition 9 and Definition 6 can be replaced by a minimum.

II. FINITE BLOCKLENGTH CONVERSES

In this section we generalise the proof of the PPV classical converse (see Section III.E of [6]), to obtain the following bounds, which are analogous to Theorem 27 of [6]. To state them, let us first introduce two quantities.

Definition 12.

$$[\rho; \mathcal{E}]_{\tilde{A}B} := \mathcal{E}_{B|A}[\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}] = \rho_{\tilde{A}}^{\frac{1}{2}*} \mathcal{E}_{B|A}[\Phi_{\tilde{A}A}] \rho_{\tilde{A}}^{\frac{1}{2}*}$$

Definition 13. By analogy with the relationship between the mutual information and relative entropy, we define:

$$I_{\epsilon}^{\Omega}(\mathcal{E}; \rho) := \min_{\sigma} D_{\epsilon}^{\Omega}([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \sigma_B). \quad (20)$$

Theorem 14 (Entanglement-assisted converse).

$$\log M_{\epsilon}^{\mathbf{E}}(\mathcal{E}_{B|A}, \rho_A) \leq I_{\epsilon}(\mathcal{E}_{B|A}, \rho_A), \text{ and so} \quad (21)$$

$$\log M_{\epsilon}^{\mathbf{E}}(\mathcal{E}_{B|A}) \leq \max_{\rho_A} I_{\epsilon}(\mathcal{E}_{B|A}, \rho_A). \quad (22)$$

Theorem 15 (Unassisted converse).

$$\log M_{\epsilon}(\mathcal{E}_{B|A}, \rho_A) \leq I_{\epsilon}^{\mathbf{L}}(\mathcal{E}_{B|A}, \rho_A), \text{ and so} \quad (23)$$

$$\log M_{\epsilon}(\mathcal{E}_{B|A}) \leq \max_{\rho_A} I_{\epsilon}^{\mathbf{L}}(\mathcal{E}_{B|A}, \rho_A). \quad (24)$$

Corollary 16. From the hierarchy of operations, it follows immediately that

$$\log M_{\epsilon}(\mathcal{E}_{B|A}, \rho_A) \leq I_{\epsilon}^{\mathbf{PPT}}(\mathcal{E}_{B|A}, \rho_A), \text{ and so} \quad (25)$$

$$\log M_{\epsilon}(\mathcal{E}_{B|A}) \leq \max_{\rho_A} I_{\epsilon}^{\mathbf{PPT}}(\mathcal{E}_{B|A}, \rho_A). \quad (26)$$

Just as in [6], these results are a consequence of a more general ‘meta-converse’, which relates the probability of correctly sending information over the channel to the problem of hypothesis testing. To establish this, we first prove (using the notation established in Definition 1)

Proposition 17. *From any entanglement-assisted code \mathcal{Z} and source \mathcal{S} , such that the average input state is ρ_A , one can construct a test $T_{\tilde{A}B} \in \mathbf{T}(\tilde{A}, B)$ such that*

$$\Pr(\hat{W} = W | \mathcal{E}, \mathcal{Z}, \mathcal{S}) = \text{Tr}[\rho; \mathcal{E}]_{\tilde{A}B} T_{\tilde{A}B}.$$

Furthermore, if \mathcal{Z} is unassisted, then $T_{\tilde{A}B} \in \mathbf{T}^{\mathbf{L}}(\tilde{A}, B)$.

Proof. Since it is always possible to augment A_E to $A_E' A_E$, take $\mathcal{C}(w)_{A|A_E' A_E}[\cdot] := \mathcal{C}(w)_{A|A_E}[\text{Tr}_{A_E'}(\cdot)]$ and $\Psi_{A_E' A_E B_E}$ a purification of $\Psi_{A_E B_E}$, we can assume that $\Psi_{A_E B_E}$ is pure.

Let the isometry $U(w)_{GA \leftarrow A_E}$ be the Stinespring representation of the encoding map $\mathcal{C}(w)_{A|A_E}$, where G is the discarded environment system. In fact, we can take $A_E = GA$ so that $U(w)_{GA \leftarrow GA} =: U(w)_{GA}$ is a unitary, and a complete $\mathcal{C}(w)_{A|A_E}[\cdot] = \text{Tr}_G \mathfrak{U}(w)[\cdot]$ where $\mathfrak{U}(w)[\cdot] := U(w)_{GA}[\cdot] U(w)_{GA}^{\dagger}$. Finally, there is no

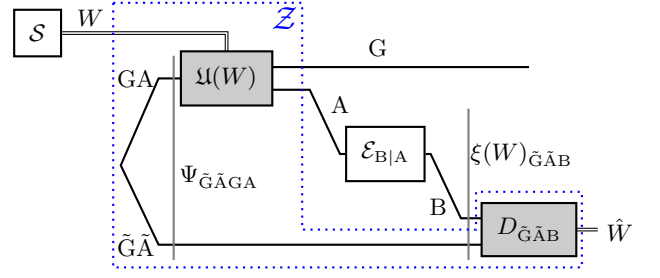


FIG. 4: Reformulation of the protocol of Fig. 2.

loss of generality in demanding that $B_E = \tilde{G}\tilde{A} \cong GA$. This reformulation of the protocol of Fig. 2 is illustrated in Fig. 4.

First note that

$$U(w)_{GA} \Psi_{GA \tilde{G}\tilde{A}} U(w)_{GA}^{\dagger} \quad (27)$$

$$= M(w)_{GA} \Phi_{GA \tilde{G}\tilde{A}} M(w)_{GA}^{\dagger} \quad (28)$$

where $M(w)_{GA} := U(w)_{GA} \Psi_{GA}^{\frac{1}{2}}$, and that

$$\rho(w)_{GA} = M(w)_{GA} M(w)_{GA}^{\dagger} \quad (29)$$

is the state of GA after encoding if the message $W = w$. Referring to the diagram, we see that

$$\Pr(\hat{W} = \hat{w} | W = w, \mathcal{E}, \mathcal{Z}) = \text{Tr} D(\hat{w})_{\tilde{G}\tilde{A}B} \xi(w)_{\tilde{G}\tilde{A}B}$$

where

$$\xi(w)_{\tilde{G}\tilde{A}B} = \text{Tr}_G \mathcal{E}_{B|A} [M(w)_{GA} \Phi_{GA \tilde{G}\tilde{A}} M(w)_{GA}^{\dagger}] \quad (30)$$

$$= \text{Tr}_G M(w)_{\tilde{G}\tilde{A}}^T \Phi_{\tilde{G}\tilde{A}} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}] M(w)_{\tilde{G}\tilde{A}}^* \quad (31)$$

$$= M(w)_{\tilde{G}\tilde{A}}^T \mathbb{1}_{\tilde{G}} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}] M(w)_{\tilde{G}\tilde{A}}^*. \quad (32)$$

Here we have used the easily verified fact that, for any linear operator M_A on \mathcal{H}_A , $M_A \Phi_{\tilde{A}A} = M_A^T \Phi_{\tilde{A}A}$. The probability of successful decoding is

$$\Pr(\hat{W} = W | \mathcal{E}, \mathcal{Z}, \mathcal{S}) \quad (33)$$

$$= \sum_{w=1}^M \mathcal{S}(w) \text{Tr} D(w)_{\tilde{G}\tilde{A}B} \xi(w)_{\tilde{G}\tilde{A}B} \quad (34)$$

$$= \text{Tr} \mathbb{1}_{\tilde{G}} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}] R_{\tilde{G}\tilde{A}B} \quad (35)$$

where

$$R_{\tilde{G}\tilde{A}B} := \sum_{w=1}^M \mathcal{S}(w) M(w)_{\tilde{G}\tilde{A}}^* D(w)_{\tilde{G}\tilde{A}B} M(w)_{\tilde{G}\tilde{A}}^T. \quad (36)$$

Since $R_{\tilde{G}\tilde{A}B}$ is given by a completely positive map (with Kraus operators $\sqrt{\mathcal{S}(w)} M(w)_{\tilde{G}\tilde{A}}^*$) acting on $D_{\tilde{G}\tilde{A}B}$, which satisfies $D_{\tilde{G}\tilde{A}B} \leq \mathbb{1}_{\tilde{G}\tilde{A}B}$, we have $0 \leq R_{\tilde{G}\tilde{A}B} \leq \rho_{\tilde{G}\tilde{A}}^* \mathbb{1}_B$, where $\rho_{GA} = \sum_{w=1}^M \mathcal{S}(w) \rho(w)_{GA}$ is the average state of GA after encoding. Therefore,

$$T_{\tilde{A}B} := \rho_{\tilde{A}}^{-\frac{1}{2}*} R_{\tilde{A}B} \rho_{\tilde{A}}^{-\frac{1}{2}*} \quad (37)$$

satisfies $0 \leq T_{\tilde{A}B} \leq \mathbb{1}_{\tilde{A}B}$, and

$$\Pr(\hat{W} = W | \mathcal{E}, \mathcal{Z}, \mathcal{S}) = \text{Tr}_{\tilde{A}B} \rho_{\tilde{A}}^{\frac{1}{2}*} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}] \rho_{\tilde{A}}^{\frac{1}{2}*} T_{\tilde{A}B} \quad (38)$$

$$= \text{Tr}_{\tilde{A}B} [\rho; \mathcal{E}]_{\tilde{A}B} T_{\tilde{A}B} \quad (39)$$

as promised.

As noted in the caption for Fig. 2, any *unassisted* quantum code corresponds to restricting Bob's decoding measurement to the output system of the channel, so that

$$\forall w \in \{1, \dots, M\} : D(w)_{\tilde{G}\tilde{A}B} = \mathbb{1}_{\tilde{G}\tilde{A}} D(w)_B.$$

Substituting this into (36), we see that

$$T_{\tilde{A}B} = \sum_{w=1}^M E(w)_{\tilde{A}} D(w)_B$$

where the positive operators

$$E(w)_{\tilde{A}} := S(w)(\rho_{\tilde{A}}^*)^{-\frac{1}{2}} (\text{Tr}_{\tilde{G}} \rho(w)_{\tilde{G}\tilde{A}}^*) (\rho_{\tilde{A}}^*)^{-\frac{1}{2}}$$

satisfy

$$\sum_{w=1}^M E(w)_{\tilde{A}} = \rho_{\tilde{A}}^{-\frac{1}{2}*} \rho_{\tilde{A}}^* \rho_{\tilde{A}}^{-\frac{1}{2}*} \leq \mathbb{1}.$$

Letting $E(0)_{\tilde{A}} := \mathbb{1} - \rho_{\tilde{A}}^{-\frac{1}{2}*} \rho_{\tilde{A}}^* \rho_{\tilde{A}}^{-\frac{1}{2}*}$, the operators $E(0)_{\tilde{A}}, \dots, E(M)_{\tilde{A}}$ constitute a POVM, and so, for an unassisted code, $T_{\tilde{A}B}$ is a *local* test i.e. $T_{\tilde{A}B} \in \mathbf{T}^L(\tilde{A}, B)$. \square

A quantum generalisation of the ‘meta-converse’ Theorem 26 of [6], is now straightforward:

Proposition 18 (Meta-converse). *Let \mathcal{Z} be an entanglement-assisted code which, when used with \mathcal{S} , induces the average input state ρ_A , and which has success probability*

$$\Pr(\hat{W} = W | \mathcal{E}^{(i)}, \mathcal{Z}, \mathcal{S}) = 1 - \epsilon_i,$$

when used with channel use $\mathcal{E}_{B|A}^{(i)}$. Consider the hypothesis testing problem where H_i asserts that the state of $\tilde{A}B$ is $[\rho; \mathcal{E}^{(i)}]_{\tilde{A}B}$. If we accept H_0 when the test constructed from $(\mathcal{Z}, \mathcal{S})$ as in Proposition 17 accepts, then

$$\beta = \text{Tr}[\rho; \mathcal{E}^{(1)}]_{\tilde{A}B} T_{\tilde{A}B} = 1 - \epsilon_1$$

and

$$1 - \alpha = \text{Tr}[\rho; \mathcal{E}^{(0)}]_{\tilde{A}B} T_{\tilde{A}B} = 1 - \epsilon_0.$$

Therefore,

$$\beta_{\epsilon_0}([\rho; \mathcal{E}^{(0)}]_{\tilde{A}B} \| [\rho; \mathcal{E}^{(1)}]_{\tilde{A}B}) \leq 1 - \epsilon_1.$$

Furthermore, if constraints on the code \mathcal{Z} mean that $T_{\tilde{A}B}$ is guaranteed to belong to some class of tests $\mathbf{T}^\Omega(A, B)$, then the (potentially more stringent) bound

$$\beta_{\epsilon_0}^\Omega([\rho; \mathcal{E}^{(0)}]_{\tilde{A}B} \| [\rho; \mathcal{E}^{(1)}]_{\tilde{A}B}) \leq 1 - \epsilon_1 \quad (40)$$

also applies.

If $\mathcal{E}_{B|A}^{(1)}$ has $\mathcal{E}_{B|A}^{(1)}[\rho_A] = \sigma_B$ for all ρ_A , and $\mathcal{S} = \mathcal{S}_M$ (i.e. the M messages are equiprobable), then it is easily verified that $1 - \epsilon_1 = 1/M$ (in fact any other value would imply that communication is possible in the absence of a channel.)

Setting $\mathcal{E}_{B|A}^{(0)} = \mathcal{E}_{B|A}$, and maximising over σ_B , we see that any (M, ϵ, ρ_A) code whose corresponding test belongs to $\mathbf{T}^\Omega(\tilde{A}, B)$ must satisfy

$$\max_{\sigma_B} \beta_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B) \leq 1/M. \quad (41)$$

For entanglement-assisted codes, rearranging this and using Definition 6 gives us

$$\log M_\epsilon^\mathbf{E}(\mathcal{E}_{B|A}, \rho_A) \leq \min_{\sigma_B} D([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B)$$

(Theorem 14) and for unassisted codes, we can write the stronger,

$$\log M_\epsilon(\mathcal{E}_{B|A}, \rho_A) \leq \min_{\sigma_B} D^\mathbf{L}([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_A^* \sigma_B)$$

(which is Theorem 15).

III. PROPERTIES OF THE BOUNDS

The results of Polyanskiy [7], show that when constrained to classical states ρ_A and classical channels $\mathcal{E}^{(0)}$, $\mathcal{E}^{(1)}$, the quantity $\beta_\epsilon([\rho; \mathcal{E}^{(0)}]_{\tilde{A}B} \| [\rho; \mathcal{E}^{(1)}]_{\tilde{A}B})$ is jointly convex in ρ_A and ϵ . Here we generalise the proof to quantum states and channels, even for restricted measurements $\Omega = \mathbf{PPT}$ or $\mathbf{LC1}$.

Theorem 19. *For any CPTP maps $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(1)}$, and class of bipartite operations Ω which contains $\mathbf{LC1}$, the function*

$$f^\Omega(\epsilon, \rho) := \beta_\epsilon^\Omega([\mathcal{E}^{(0)}; \rho]_{\tilde{A}B} \| [\mathcal{E}^{(1)}; \rho]_{\tilde{A}B})$$

is jointly convex in ϵ and ρ .

Proof. First, note that $f^\Omega(\epsilon, \rho) = \tilde{f}^\Omega(\epsilon, \rho^*)$ where

$$\tilde{f}^\Omega(\epsilon, \rho) := \beta_\epsilon(\mathcal{E}_{\tilde{A}B}^{(0)}[\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}], \mathcal{E}_{\tilde{A}B}^{(1)}[\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}])$$

so it will suffice to show the joint convexity property for \tilde{f}^Ω . Suppose we have $\epsilon_j, \rho(j)_{\tilde{A}}$, and λ_j for $j = 1, \dots, m$, with $\sum_{j=1}^m \lambda_j = 1$. Let $\epsilon = \sum_{j=1}^m \lambda_j \epsilon_j$ and $\rho = \sum_{j=1}^m \lambda_j \rho(j)_{\tilde{A}}$.

Let T_j be a test in \mathbf{T}^Ω that achieves $\tilde{f}^\Omega(\epsilon_j, \rho_j)$, and let

$$T_{\tilde{A}B} = \rho_{\tilde{A}}^{-\frac{1}{2}} \left(\sum_{j=1}^m \lambda_j \rho(j)_{\tilde{A}}^{\frac{1}{2}} T(j)_{\tilde{A}B} \rho(j)_{\tilde{A}}^{\frac{1}{2}} \right) \rho_{\tilde{A}}^{-\frac{1}{2}}.$$

This test can be implemented in the following way: First Alice performs an measurement with measurement operator $\sqrt{\lambda_j} \rho(j)_{\tilde{A}}^{\frac{1}{2}} \rho_{\tilde{A}}^{-\frac{1}{2}}$ for outcome $j = 1, \dots, m$, and an

operator for outcome $j = 0$ to make the measurement complete.

Now, based on the outcome of this measurement j (which is classically communicated from Alice to Bob) Alice and Bob perform the measurement to implement the test $T(j)_{\tilde{A}B}$ (they accept H_0 when the this test $T(j)_{\tilde{A}B}$ would have).

Since this implementation uses only one-way classical communication followed by operations in class Ω , which, by hypothesis, contains **LC1**, this test is also in $\mathbf{T}^\Omega(\tilde{A}, B)$. The type-I error α for $T_{\tilde{A}B}$ is

$$\begin{aligned} & \text{Tr} \mathcal{E}_{\tilde{A}B}^{(0)} [\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}] T_{\tilde{A}B} \\ &= \text{Tr} \sum_{j=1}^m \lambda_j \rho(j)_{\tilde{A}}^{\frac{1}{2}} T(j)_{\tilde{A}B} \rho(j)_{\tilde{A}}^{\frac{1}{2}} \mathcal{E}_{\tilde{A}B}^{(0)} [\Phi_{\tilde{A}A}] \\ &= \sum_{j=1}^m \lambda_j \text{Tr} T(j)_{\tilde{A}B} \mathcal{E}_{\tilde{A}B}^{(0)} [\rho(j)_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho(j)_{\tilde{A}}^{\frac{1}{2}}] \\ &\geq \sum_{j=1}^m \lambda_j (1 - \epsilon_j) = 1 - \epsilon. \end{aligned}$$

Similarly, the type-II error β is

$$\text{Tr} T_{\tilde{A}B} \mathcal{E}_{\tilde{A}B}^{(1)} [\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}] = \sum_j \lambda_j \tilde{f}^\Omega(\epsilon_j, \rho(j)_{\tilde{A}}).$$

Therefore $\tilde{f}^\Omega(\epsilon, \rho_{\tilde{A}}) \leq \sum_j \lambda_j \tilde{f}^\Omega(\epsilon_j, \rho(j)_{\tilde{A}})$, as claimed. \square

Corollary 20. $I_\epsilon(\mathcal{E}, \rho)$ is jointly concave in ϵ and ρ .

A. Classical channels

Let $\mathcal{C}_{A|A}$ and $\mathcal{C}_{B|B}$ denote the completely dephasing operations in the classical bases for A and B, respectively. If $\mathcal{E}_{B|A}$ is classical then $\mathcal{E}_{B|A} \circ \mathcal{C}_{A|A} = \mathcal{E}_{B|A}$, and $\mathcal{C}_{B|B} \circ \mathcal{E}_{B|A} = \mathcal{E}_{B|A}$, and we can therefore restrict the minimization over average input states in our bounds to states ρ_A which are diagonal in the classical basis, thus

$$\rho_A = \sum_x p(x) |x\rangle\langle x|_A.$$

Furthermore, since the state $[\rho; \mathcal{E}]_{\tilde{A}B}$ is invariant under the operation $\mathcal{C}_{B|B} \in \mathbf{ops}^L(\tilde{A} \rightarrow \tilde{A}, B \rightarrow B)$, for any class Ω containing **L** we have (by Theorem 7)

$$D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \|\rho_{\tilde{A}}^* \sigma_B) \quad (42)$$

$$\geq D_\epsilon^\Omega(\mathcal{C}_{B|B}[\rho; \mathcal{E}]_{\tilde{A}B} \|\mathcal{C}_{B|B} \rho_{\tilde{A}}^* \sigma_B) \quad (43)$$

$$= D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \|\rho_{\tilde{A}}^* \sigma'_B) \quad (44)$$

where σ'_B is diagonal in the classical basis of B. Therefore, we can also restrict the optimisation over σ_B to classical states. Therefore, Theorems 14 and 15 both reduce to the PPV converse for finite alphabets when the channel is classical.

B. Semidefinite programs

Looking at Proposition 9 it is clear that β_ϵ is given by the solution of a semidefinite program (SDP). What is less obvious, is that $I_\epsilon(\mathcal{E}_{B|A}, \rho_A)$ and its maximisation over ρ_A can also be formulated as (simple functions of the solutions to) SDPs. We have

$$I_\epsilon(\mathcal{E}_{B|A}, \rho_A) = \min_{\sigma_B} D_\epsilon([\rho; \mathcal{E}]_{\tilde{A}B} \|\rho_{\tilde{A}}^* \sigma_B) \quad (45)$$

$$= -\log \max_{\sigma_B} \min_{T_{\tilde{A}B}} \text{Tr}_{\tilde{A}B} \rho_{\tilde{A}}^* \sigma_B T_{\tilde{A}B} \quad (46)$$

subject to the constraints

$$\text{Tr}[\rho; \mathcal{E}]_{\tilde{A}B} T_{\tilde{A}B} \geq 1 - \epsilon, \quad 0 \leq T_{\tilde{A}B} \leq \mathbb{1}_{\tilde{A}B}. \quad (47)$$

By von Neumann's minimax theorem, the order of the maximization and minimization in the equation (46) can be reversed. Then, using $\max_{\sigma_B \in \mathcal{D}(\mathcal{H}_B)} \text{Tr}_{\tilde{A}B} \rho_{\tilde{A}}^* \sigma_B T_{\tilde{A}B} = \max_{\sigma_B \in \mathcal{D}(\mathcal{H}_B)} \text{Tr}_B \sigma_B (\text{Tr}_A \rho_{\tilde{A}}^* T_{\tilde{A}B}) = \|\text{Tr}_A \rho_{\tilde{A}}^* T_{\tilde{A}B}\|_\infty = \min\{\lambda : \lambda \mathbb{1}_B \geq \text{Tr}_A \rho_{\tilde{A}}^* T_{\tilde{A}B}\}$ we have

$$I_\epsilon(\mathcal{E}_{B|A}, \rho_A) = -\log \min_{T, \lambda} \lambda \quad (48)$$

$$\begin{aligned} & \text{subject to} \\ & \lambda \mathbb{1}_B \geq \text{Tr}_A \rho_{\tilde{A}}^* T_{\tilde{A}B} \end{aligned} \quad (49)$$

and the constraints (47). With the change of variables $R_{\tilde{A}B} := \rho_{\tilde{A}}^{\frac{1}{2}} T_{\tilde{A}B} \rho_{\tilde{A}}^{\frac{1}{2}}$, this is equivalent to

Proposition 21 (Primal SDP).

$$I_\epsilon(\mathcal{E}_{B|A}, \rho_A) = -\log \min_{T, \lambda} \lambda \quad (50)$$

subject to

$$\text{Tr}_A R_{\tilde{A}B} \leq \lambda \mathbb{1}_B, \quad (51)$$

$$\text{Tr} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}] R_{\tilde{A}B} \geq 1 - \epsilon, \quad (52)$$

$$R_{\tilde{A}B} \leq \rho_{\tilde{A}}^* \mathbb{1}_B, \quad (53)$$

$$R_{\tilde{A}B} \geq 0. \quad (54)$$

Since the constraints on ρ_A are semidefinite, the bound

$$\max_{\rho_A \in \mathcal{D}(\mathcal{H}_A)} I_\epsilon(\mathcal{E}_{B|A}, \rho_A)$$

from Theorem 14 is also a semidefinite program.

Remark 22. The constraint $0 \leq \text{t}_B T_{\tilde{A}B} \leq \mathbb{1}$ (where t_B is the transpose map on system B) is equivalent to

$$0 \leq \text{t}_B R_{\tilde{A}B} \leq \rho_{\tilde{A}}^* \mathbb{1}_B.$$

Because the transpose map is linear, adding these constraints on $R_{\tilde{A}B}$ to the primal SDP above yields an SDP for $I_\epsilon^{\mathbf{PPT}}(\mathcal{E}_{B|A}, \rho_A)$.

Associating operators $F_{\tilde{A}B}$ and G_B with constraints (53) and (51), and a real multiplier μ with (52) yields

the Lagrangian

$$\begin{aligned} & \lambda + \text{Tr} G_B (\text{Tr}_{\tilde{A}} R_{\tilde{A}B} - \lambda \mathbb{1}_B) \\ & + \text{Tr} F_{\tilde{A}B} (R_{\tilde{A}B} - \rho_{\tilde{A}}^* \otimes \mathbb{1}_B) \\ & + \mu (1 - \epsilon - \text{Tr} R_{\tilde{A}B} \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}]) \\ & = \text{Tr} R_{\tilde{A}B} (\mathbb{1}_{\tilde{A}} \otimes G_B + F_{\tilde{A}B} - \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}]) \\ & + \lambda (1 - \text{Tr} G_B) + (1 - \epsilon) \mu. \end{aligned}$$

from which one can derive the dual SDP. Below, we show that the optimal value of this dual SDP is equal to the optimal value of the primal.

Proposition 23 (Dual SDP).

$$I_\epsilon(\mathcal{E}_{B|A}, \rho_A) = -\log \max(1 - \epsilon) \mu - \text{Tr} F_{\tilde{A}B} \rho_{\tilde{A}}^* \quad (55)$$

subject to

$$\mathbb{1}_{\tilde{A}} \otimes G_B + F_{\tilde{A}B} \geq \mu \mathcal{E}_{B|A} [\Phi_{\tilde{A}A}], \quad (56)$$

$$\text{Tr} G_B \leq 1, \quad (57)$$

$$G_B, F_{\tilde{A}B}, \mu \geq 0. \quad (58)$$

Proof. For sufficiently large a , the point given by $G_B = \mathbb{1}_B / (2 \dim(B))$, $F_{\tilde{A}B} = a \mathbb{1}_{\tilde{A}B}$, and any $\mu > 0$ strictly satisfies the dual constraints (56-58), so the dual SDP is strictly feasible, and therefore its solution is equal to the primal solution (see Theorem 3.1 of [14]). \square

The maximisation of (55) over states ρ_A of \tilde{A} can also be formulated as an SDP, in a similar way to the primal.

C. Comparison with Wang–Renner

In our notation, the Wang–Renner converse states that for c-q channels with finite input alphabet A and output states $\tau(x)_B$ for $x \in A$

$$\log M_\epsilon \leq \sup_p D_\epsilon(\tau_{CB} \| \tau_C \tau_B)$$

where C is a system of dimension $|A|$ and $\tau_{CB} := \sum_{x \in A} p(x) |x\rangle\langle x|_C \otimes \tau(x)_B$. To apply their converse to general channels, one notes that any (unassisted) code for a general quantum channel, induces a c-q channel by its specification of the input states used in the code. Together with the choice of p the yields a c-q state

$$\tau_{CB} = \sum_x p(x) |x\rangle\langle x|_C \otimes \mathcal{E}_{B|A}[\rho(x)_A]$$

Optimising the Wang–Renner bound over all choices of input states and distributions p such that the average channel input to the quantum channel is ρ_A , one obtains

$$M_\epsilon(\mathcal{E}_{B|A}) \leq \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A)$$

where

Definition 24.

$$\chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) := \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D_\epsilon(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A).$$

and

Definition 25. Let $\text{Ens}(\rho_A)$ be the set of all states of the form

$$\sum_{k=1}^{\dim(C)} p_k |k\rangle\langle k|_C \otimes \rho(k)_A$$

where C is some finite dimensional system (acting as a classical register) and where $p_k \geq 0$, $\sum_k p_k = 1$, $\rho(k)_A \in \mathcal{D}(\mathcal{H}_A)$, and

$$\sum_k p_k \rho(k)_A = \rho_A.$$

This notation is motivated by the fact that the Holevo bound [15] is given by

$$\chi(\mathcal{E}_{B|A}, \rho_A) = \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A),$$

where D is the usual quantum relative entropy.

Proposition 26.

$$\chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) = D_\epsilon^{\text{LC1}}([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \mathcal{E}_{B|A}[\rho_A])$$

Proof. Let Alice’s measurement have the POVM elements $F(k)_{\tilde{A}}$ where k labels the outcome which she sends to Bob. There is no loss of generality in having Alice perform before Bob does anything, and storing the outcome in a classical register C to which Bob has access.

Letting $p(k) := \text{Tr} \rho_{\tilde{A}}^* F(k)_{\tilde{A}}$ be the probability of outcome k and $\rho(k)_A = \text{Tr}_{\tilde{A}} (\rho_{\tilde{A}}^{\frac{1}{2}} \Psi_{\tilde{A}A} \rho_{\tilde{A}}^{\frac{1}{2}}) F_{\tilde{A}}(k)$, under hypothesis 0 the state of CB is $\mathcal{E}_{B|A} \eta_{CA}$, where

$$\eta_{CA} = \sum_k p(k) |k\rangle\langle k|_C \otimes \rho(k)_A,$$

while under hypothesis 1 the state is $\eta_C \mathcal{E}_{B|A} \eta_A$. Clearly $\eta_A = \sum_k p(k) \rho(k)_A = \rho_A$. Therefore,

$$\begin{aligned} & D_\epsilon^{\text{LC1}}([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \mathcal{E}_{B|A}[\rho_A]) \\ & = \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D_\epsilon(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A) \\ & = \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A). \end{aligned}$$

\square

Corollary 27. From the above proposition and the definitions of the quantities involved, the inequalities

$$I_\epsilon^{\text{L}}(\mathcal{E}_{B|A}, \rho_A) \leq I_\epsilon^{\text{LC1}}(\mathcal{E}_{B|A}, \rho_A) \leq \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A)$$

follow immediately.

D. Asymptotics

It was already noted in [2] that the (asymptotically tight) Holevo bound on unassisted codes can be recovered from an asymptotic analysis of the Wang-Renner bound, which our bounds on unassisted codes subsume (in fact, an argument of [5] can be used to show that the converse part of the HSW theorem [16, 17] can also be derived).

For entanglement-assisted coding over memoryless quantum channels, Shannon's noisy channel coding theorem has a beautiful generalisation due to Bennett, Shor, Smolin and Thapliyal:

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log M_\epsilon^{\mathbf{E}}((\mathcal{E}^{\otimes n})_{B^n|A^n}) = \max_{\rho_A} I(\mathcal{E}_{B|A}; \rho_A) \quad (59)$$

where

$$I(\mathcal{E}_{B|A}; \rho_A) := S(\rho_A) + S(\mathcal{E}_{B|A}(\rho_A)) - S([\rho; \mathcal{E}]_{\tilde{A}B})$$

is the *quantum mutual information* between systems \tilde{A} and B when the state of $\tilde{A}B$ is $[\rho; \mathcal{E}]_{\tilde{A}B}$. As noted, for classical channels Theorem 14 reduces to the Theorem 27 of [6]. In section III.G of [6] it is shown how to derive a Fano-type converse from this theorem. The derivation and result generalise perfectly to the entanglement-assisted codes for quantum channels: As usual, the binary entropy is $h(p) := -(1-p)\log(1-p) - p\log p$, and the binary relative entropy is

$$d(p||q) := D((p, 1-p)|| (q, 1-q)) \quad (60)$$

$$= p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \quad (61)$$

$$\geq p \log \frac{1}{q} + h(p). \quad (62)$$

By the data processing inequality for quantum relative entropy under CPTP maps, and (62)

$$D(\rho_0||\rho_1) \geq d(1-\epsilon||\beta_\epsilon(\rho_0, \rho_1)) \quad (63)$$

$$\geq (1-\epsilon) \log \frac{1}{\beta_\epsilon(\rho_0, \rho_1)} + h(\epsilon). \quad (64)$$

Therefore,

$$D_\epsilon(\rho_0||\rho_1) \leq (D(\rho_0||\rho_1) - h(\epsilon))/(1-\epsilon). \quad (65)$$

Setting $\rho_0 = \mathcal{E}_{B|A}[\rho_A^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_A^{\frac{1}{2}}]$ and $\rho_1 = \rho_A^* \sigma_B$, and minimizing over σ_B yields

Lemma 28.

$$I_\epsilon(\mathcal{E}_{B|A}, \rho_A) \leq (I(\mathcal{E}_{B|A}; \rho_A) - h(\epsilon))/(1-\epsilon) \quad (66)$$

The converse part of this theorem is easily derived from the previous lemma, which tells us that

$$\frac{1}{n} \log M_\epsilon^{\mathbf{E}}(\mathcal{E}) \leq \frac{1}{n} \max_{\rho_A} \frac{I(\mathcal{E}^{\otimes n}; \rho_{A^n}) - h(\epsilon)}{1-\epsilon}. \quad (67)$$

In [18] Adami and Cerf show that

$$I(\mathcal{E}_{B_1|A_1}^{(1)} \mathcal{E}_{B_2|A_2}^{(2)}, \rho_{A_1 A_2}) = I(\mathcal{E}_{B_1|A_1}^{(1)}, \rho_{A_1}) + I(\mathcal{E}_{B_2|A_2}^{(2)}, \rho_{A_2}),$$

so we have

$$\max_{\rho_{A^n}} I((\mathcal{E}^{\otimes n})_{B^n|A^n}, \rho_{A^n}) \leq n \max_{\rho_A} I(\mathcal{E}_{B|A}, \rho_A),$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log M_\epsilon((\mathcal{E}^{\otimes n})_{B^n|A^n}) \leq \frac{1}{1-\epsilon} \max_{\rho} I(\mathcal{E}_{B|A}; \rho_A), \quad (68)$$

and taking the limit $\epsilon \rightarrow 0$ completes the proof.

E. Using symmetries

We now show how symmetries of $\mathcal{E}_{B|A}$ can be used to simplify the computation of $I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A)$. For classical channels, analogous results were obtained in [7] (but note that [7] also deals with infinite input/output alphabets) and similar ideas discussed in [8].

Suppose that there is a group G with an action g_A ($g_{\tilde{A}}$) on states of A (and of \tilde{A}) given by

$$g_A \tau_A = U(g)_A \tau_A U(g)_A^\dagger,$$

and an action g_B on states of B

$$g_B \tau_B = V(g)_B \tau_B V(g)_B^\dagger,$$

where U and V are unitary representations of G , such that

$$\forall g \in G : \mathcal{E}_{B|A}(g_A \rho_A) = g_B \mathcal{E}_{B|A}(\rho_A).$$

Proposition 29. For any $\Omega \supseteq \mathbf{L}$ and for all $g \in G$,

$$D_\epsilon^\Omega([\mathcal{E}; g_A \rho]_{\tilde{A}B} \| g_{\tilde{A}}^* \rho_{\tilde{A}}^* g_B \sigma_B) = D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \sigma_B) \quad (69)$$

and $I_\epsilon^\Omega(\mathcal{E}_{B|A}, g_A \rho_A) = I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A)$.

Proof. The first claim follows from

$$\begin{aligned} & [\mathcal{E}; g_A \rho]_{\tilde{A}B} \\ &= \mathcal{E}_{B|A}[U(g)_A \rho_A^{\frac{1}{2}} U(g)_A^\dagger \Phi_{\tilde{A}A} U(g)_A \rho_A^{\frac{1}{2}} U(g)_A^\dagger] \\ &= \mathcal{E}_{B|A}[U(g)_A \rho_A^{\frac{1}{2}} U(g)_A^* \Phi_{\tilde{A}A} U(g)_A^T \rho_A^{\frac{1}{2}} U(g)_A^\dagger] \\ &= V(g)_B U(g)_A^* \mathcal{E}_{B|A}[\rho_A^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_A^{\frac{1}{2}}] V(g)_B U(g)_A^T \\ &= g_{\tilde{A}}^* g_B [\rho; \mathcal{E}]_{\tilde{A}B}, \end{aligned}$$

the fact that $g_{\tilde{A}}^* g_B, (g^{-1})_{\tilde{A}}^* (g^{-1})_B \in \mathbf{L}(\tilde{A} \rightarrow \tilde{A}, B \rightarrow B) \subseteq \Omega$, and Corollary 8. We use this to prove the second claim thus:

$$\begin{aligned} I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A) &= \min_{\sigma_B} D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \sigma_B) \\ &= D_\epsilon^\Omega([\rho; \mathcal{E}]_{\tilde{A}B} \| \rho_{\tilde{A}}^* \sigma_B^0) \\ &= D_\epsilon^\Omega([\mathcal{E}; g_A \rho]_{\tilde{A}B} \| g_{\tilde{A}}^* \rho_{\tilde{A}}^* g_B \sigma_B^0) \\ &\geq \min_{\sigma_B} D_\epsilon^\Omega([\mathcal{E}; g_A \rho]_{\tilde{A}B} \| g_{\tilde{A}}^* \rho_{\tilde{A}}^* \sigma_B) \\ &= I_\epsilon^\Omega(\mathcal{E}_{B|A}, g_A \rho_A). \end{aligned}$$

Since g has an inverse in G , the reverse inequality holds too. \square

Suppose that there is a Haar (G -invariant) measure μ on G , and let $\bar{\rho}_A := \int_G g_A \rho_A \mu(g)$. Then, $\bar{\rho}_A$ is invariant under the action g_A , and by Jensen's inequality and Proposition 29,

$$I_\epsilon(\mathcal{E}_{B|A}, \bar{\rho}_A) \geq \int_G I_\epsilon(\mathcal{E}_{B|A}, g_A \rho_A) \mu(g) = I_\epsilon(\mathcal{E}_{B|A}, \rho_A).$$

Therefore, the optimisation over ρ_A can be restricted to those density operators invariant under the action of g_A .

An important type of symmetry that an operation representing n channel uses may possess is permutation covariance. For example, this applies to channels of the form $\mathcal{E}^{\otimes n}$.

For any element π of the symmetric group S_n , and n -partite system $Q^n := Q_1 Q_2 \dots Q_n$ consisting of n isomorphic systems Q_j , let $\pi_{Q^n} \in \mathbf{ops}(Q^n \rightarrow Q^n)$ denote the unitary operation which permutes the n systems.

An operation $\mathcal{E}_{B^n|A^n} \in \mathbf{ops}(A^n \rightarrow B^n)$ is permutation covariant if

$$\mathcal{E}_{B^n|A^n} \circ \pi_{A^n} = \pi_{B^n} \circ \mathcal{E}_{B^n|A^n}.$$

Suppose that, in addition to permutation invariance of the n uses, each *use* of the channel is G -covariant in the sense that

$$\mathcal{E}_{B^n|A^n} \circ \mathbf{g}_{A^n} = \mathbf{g}_{B^n} \circ \mathcal{E}_{B^n|A^n}$$

where $\mathbf{g}_{A^n} = g_{A_1}^{(1)} g_{A_2}^{(2)} \dots g_{A_n}^{(n)}$ is an action of $\mathbf{g} = (g^{(1)}, g^{(2)}, \dots, g^{(n)}) \in G^{\times n}$, where we demand that the individual g_{A_j} are all the same, in the sense that, if $\chi_{A_i A_j}$ is the permutation which swaps systems A_i and A_j , then $\chi_{A_i A_j} \circ g_{A_j} \circ \chi_{A_i A_j} = g_{A_i}$ for all $i, j \in \{1, 2, \dots, n\}$, (and similarly for \mathbf{g}_{B^n}).

To every ordered pair (π, \mathbf{g}) where $\pi \in S_n$ and $\mathbf{g} \in G^{\times n}$, we can associate an action $(\pi, \mathbf{g})_{A^n} := \pi_{A^n} \circ \mathbf{g}_{A^n}$. Under composition these actions constitute a group, which is a semi-direct product of $G^{\times n}$ and S_n ($G^{\times n}$ being the normal subgroup) which we denote $S_n \ltimes G^{\times n}$. Defining the action of $(\pi, \mathbf{g}) \in S_n \ltimes G^{\times n}$ on states of B^n by $(\pi, \mathbf{g})_{B^n} := \pi_{B^n} \circ \mathbf{g}_{B^n}$, we have

$$\mathcal{E}_{B^n|A^n} \circ (\pi, \mathbf{g})_{A^n} = (\pi, \mathbf{g})_{B^n} \circ \mathcal{E}_{B^n|A^n}.$$

IV. EXAMPLES

Example 30 (The depolarising channel). A single use of the d -dimensional depolarising channel with parameter p and d -dimensional input and output systems A and B has the operation

$$\mathcal{D}_{B|A}[\tau_A] = (1-p)\tau_B + p\text{Tr}(\tau_B)\mu_B,$$

and for n uses the operation is

$$\mathcal{D}_{B^n|A^n}^{\otimes n} = \mathcal{D}_{B_1|A_1} \dots \mathcal{D}_{B_n|A_n},$$

which has the covariance group $S_n \ltimes \text{U}(d)^{\times n}$.

The only input and output states with the corresponding invariances are the maximally mixed states. Therefore,

$$\sup_{\rho_{A^n}} I_\epsilon(\mathcal{D}_{B^n|A^n}^{\otimes n} : \rho_{A^n}) = D_\epsilon(\phi(p)_{\tilde{A}^n B^n}^{\otimes n} \| \mu_{\tilde{A}^n} \mu_{B^n})$$

where $\phi(p)_{\tilde{A}B} := \mathcal{D}_{B|A}[\Phi_{\tilde{A}A}/d] = (1-p)\Phi_{\tilde{A}B}/d + p\mu_{\tilde{A}}\mu_B$ is an *isotropic* state. Since the arguments of D_ϵ commute, this is equivalent to a classical hypothesis test between the distributions given by the eigenvalues of the two states. In fact it is equivalent to deciding between hypotheses on the distribution of n samples of a binary variable: Hypothesis H_0 is that the samples are drawn i.i.d. with probability $(1-p) + p/d^2$ of being 0, and hypothesis H_1 is that the samples are drawn i.i.d. with probability $1/d^2$ of being 0. Preparing system $\tilde{A}_j B_j$ in state $\Phi(x_j)$ produces ρ_i when H_i is the case. Conversely, setting x_j to 0 if $\Phi(0)$ is measured on system $\tilde{A}_j B_j$ and setting it to 1 otherwise, yields H_i when the state is ρ_i . So, by Corollary 8,

$$\begin{aligned} & D_\epsilon(\phi(p)_{\tilde{A}^n B^n}^{\otimes n} \| \mu_{\tilde{A}^n} \mu_{B^n}) \\ &= D_\epsilon((\mu, 1-\mu)^{\otimes n}, (\lambda, 1-\lambda)^{\otimes n}) \end{aligned}$$

where $\mu = (1-p) + p/d^2$ and $\lambda = 1/d^2$, and Proposition 31 gives a formula for this quantity which is easy to evaluate exactly (as we have done for Fig. 1).

Proposition 31. *Let $\mu \geq \lambda$ be two probabilities.*

$$\beta_\epsilon((\mu, 1-\mu)^{\otimes n}, (\lambda, 1-\lambda)^{\otimes n}) = (1-\gamma)\beta_{\ell(\epsilon)} + \gamma\beta_{\ell(\epsilon)+1}$$

where

$$\alpha_\ell = \sum_{j=0}^{\ell-1} \binom{n}{j} \mu^j (1-\mu)^{n-j}, \quad (70)$$

$$\beta_\ell = \sum_{j=l}^n \binom{n}{j} \lambda^j (1-\lambda)^{n-j}, \quad (71)$$

$\ell(\epsilon)$ is the value of l satisfying $\alpha_\ell \leq \epsilon \leq \alpha_{\ell+1}$ and $\gamma = (\alpha - \alpha_{\ell(\alpha)})/(\alpha_{\ell(\alpha)+1} - \alpha_{\ell(\alpha)})$.

Proof. This is just optimising over the optimal (classical) hypothesis tests identified by the Neyman-Pearson lemma. The same expression is given in [6]. \square

V. STRONG CONVERSES AND CRYPTOGRAPHY

In [19] a strong converse was proven for the classical capacity of many quantum channels \mathcal{N} , including depolarising noise. In particular, it was shown that for any

rate R strictly above the capacity of such channels, there exists a $\gamma(R, \mathcal{N}) > 0$ such that the success probability of transmitting Rn uniformly random bits through $\mathcal{N}^{\otimes n}$ decreases exponentially in n

$$P_{\text{succ}}(\mathcal{N}^{\otimes n}, R) \leq 2^{-\gamma(R, \mathcal{N})n}, \quad (72)$$

where the so-called strong converse parameter is given by

$$\gamma(R, \mathcal{N}) = \max_{1 < \alpha \leq 2} \left(1 - \frac{1}{\alpha}\right) (R - \chi_{\alpha}^*(\mathcal{N})) , \quad (73)$$

with $\chi_{\alpha}^*(\mathcal{N})$ the α -Holevo quantity [19], and

$$P_{\text{succ}}(\mathcal{N}^{\otimes n}, R) = \max_{\{\rho_x\}_x, \{M_x\}_x} \frac{1}{2^{nR}} \sum_{x \in \{0,1\}^{nR}} \text{Tr} [M_x \mathcal{N}^{\otimes n}(\rho_x)]$$

where the maximisation is taken over all encodings ρ_x and decoding POVMs $\{M_x\}_x$. Note that for rates above the capacity, we thus have that for any fixed possible ϵ that $\epsilon \geq 1 - 2^{-\gamma(R, \mathcal{N})n}$. Hence, for any ϵ we can solve (73) to yield an upper bound on the rate R , and thus $\log M_{\epsilon}$, as

$$\log M_{\epsilon} \leq \frac{1 - \alpha}{\alpha} \log(1 - \epsilon) + \chi_{\alpha}^*(\mathcal{N}) \quad (74)$$

for any $\alpha \in (1, 2]$. Note, however, that the present work is of a fundamentally different nature in that we provide bounds for $\log M_{\epsilon}$ for any rate, even below the capacity.

The results of [19] had a nice application to proving security in so-called noisy-storage model of quantum cryptography [20]. In short, this model allows for the secure implementation of any two-party cryptographic task under the assumption that the adversary's quantum memory is noisy. Examples of such tasks include bit commitment and oblivious transfer, which are impossible to achieve without assumptions [21, 22]. Whereas more recent work also allows us to link the security to the entanglement cost [23] or the quantum capacity [24], we here refer to a security statement in terms of the classical capacity. Specifically, it was shown in [20] that for any c-q state ρ_{XQ} , where Alice holds X and Q is held by an adversary

$$H_{\min}(X|\mathcal{F}(Q)) \geq -\log P_{\text{succ}}(\mathcal{F}, [H_{\min}(X)]) . \quad (75)$$

That is, by understanding $H_{\min}(X)$ alone and the properties of the channel \mathcal{F} we can bound the adversaries knowledge about a string X of length k . In [20] a bound on $H_{\min}(X) \gtrsim k/2$ was obtained by an uncertainty relation for BB84 measurements that were used to generate the string X , up to a security error $\tilde{\epsilon}$. Different measurements lead to higher (or lower) values of $H_{\min}(X)/k =: \hat{R}$. The strong converse for the channel $\mathcal{F} = \mathcal{N}^{\otimes n}$ was then used to bound the r.h.s. for the rate $R = (k\hat{R})/n$, as long as R exceeded the capacity.

However, up to this date no security statements were known for arbitrary \mathcal{F} , i.e., channels that may not obey a strong converse or simply structureless channels. Let

us sketch that our approach directly leads to a security statement for any \mathcal{F} that can be bounded using a semidefinite program (although it should be noted that this is not easy to evaluate if the input dimension of \mathcal{F} is large). Yet, this approach leads to security even without the use of a strong converse for *any* \mathcal{F} . More specifically, we will turn things around and ask how large we have to choose k such that sending $k\hat{R}$ bits through the channel \mathcal{F} will incur an error of at least ϵ . Intuitively, our goal is to effectively overflow the adversary's storage device \mathcal{F} . By the results of [20] we can then bound the adversary's min-entropy as $H_{\min}(X|\mathcal{F}(Q)) \geq -\log(1 - \epsilon)$. For any $\epsilon > 0$, our analysis yields a bound on $\log M_{\epsilon}$ stating that if we were to transmit more than $k\hat{R} > \log M_{\epsilon}$ bits, the error is necessarily $\epsilon + \delta > \epsilon$ for some $\delta > 0$, which yields the desired bound. As \hat{R} is determined by an uncertainty relation, we thus know how many transmissions k we have to make to obtain security.

VI. CONCLUSION

We have shown how a simple and powerful idea [6] for obtaining a finite blocklength converse for classical channels in terms of a hypothesis testing problem can be generalised to quantum channels and entanglement-assisted codes.

This generalisation has the property that a natural restriction on codes (removing entanglement-assistance) translates into a natural restriction on the tests that can be performed in the hypothesis testing problem (they must be local). This provides a strong link between the extensively studied problems of channel coding and hypothesis testing (and state discrimination) of bipartite systems under locality restrictions.

Many avenues for further work are apparent to the authors: Subsection III E invites a more thorough investigation into the extent to which symmetries can be used to simplify evaluation of the bound in various special cases, and especially in the case of general memoryless channels, where one might hope for an exponential reduction in the size of the SDP, in a quantum generalisation of [8].

While we have been able to fit the existing converse of Wang and Renner [2] precisely into our hierarchy bounds based on restricted hypothesis testing, it is not obvious to the authors what relationship exists between our converse for entanglement-assisted codes and that of Datta and Hsieh [4]. We would like to know what can be said about this, particularly in light of the achievability bound given in [4].

A limitation of our work is that we only generalise the classical bound of [6] for the case of finite input/output alphabets (as our input/output systems have finite dimension). To analyse the general case will require greater mathematical sophistication (see [7]), but would be desirable given (for example) the interest in quantum gaussian channels [25].

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